Ellipses

Standard: G-GPE #3 – Derive the equation of ellipses given the foci.

Essential Question: How do we write the equation of an ellipse?

**Ellipse** – the set of all points $(x, y)$ whose sum of the distances from two distinct points, called foci, is constant.

**Vertices** – where the line through the foci intersects the ellipse

**Major axis** – the chord joining the vertices

**Minor axis** – the chord perpendicular to the major axis at the center

**Center** – the intersection of the major and minor axis

Identify the following

- Major axis
- Minor axis
- Foci
- Vertices

Standard Form Equation of an Ellipse: with center $(h, k)$, major axis of length $2a$, & minor axis of length $2b$, where $0 < b < a$. The foci lie on the major axis $c$ units away from the center, where $c^2 = a^2 - b^2$.

**Major Axis of Horizontal**

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

**Major Axis of Vertical**

\[
\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1
\]

*The larger denominator $(a)$ determines whether the major axis is horizontal or vertical.

**Example**

1. \[
\frac{(x-5)^2}{16} + \frac{(y+4)^2}{9} = 1
\]

   - **Center:** $(5, -4)$
   - **Major Axis:** $\overline{H} \text{ or } V$
   - **Length of Major Axis:** 8
   - **Length of Minor Axis:** 6

\[
\begin{align*}
    Foci & : (5, -4) + (7, 0, 4, -4) \\
    Vertices & : (1, -4), (9, -4) \\
    Endpoints of Minor Axis & : (5, -7), (5, -1)
\end{align*}
\]

\[c = \sqrt{22.09}\]
Part 1: Graph each ellipse below.

1. \( \frac{(x + 1)^2}{4} + \frac{(y - 4)^2}{25} = 1 \)

**Center:** \((-1, 4)\)

**Major Axis:** H or \(\text{V}\)

**Length of Major Axis:** \(10\)

**Length of Minor Axis:** \(6\)

**Foci points:**\
\((-1, 4.58)\) \((-1, -4.58)\)

**Vertices:** \((-1, -1), (-1, 9)\)

**Endpoints of Minor Axis:** \((-3, 4), (1, 4)\)
2. \( \frac{(x-3)^2}{16} + (y+2)^2 = 1 \)

Center: \((3, -2)\)  
Major Axis: \(H\) or \(V\)

Length of Major Axis: 8  
Length of Minor Axis: 2\(r\)

Foci points: \((-3.87, -2)\), \((3.87, -2)\)  
Vertices: \((-1, -2)\), \((7, -2)\)

Endpoints of Minor Axis: \((3, -3)\), \((3, -1)\)

3. \( \frac{x^2}{54} + \frac{(y-8)^2}{10} = 1 \)

Center: \((0, 8)\)  
Major Axis: \(H\) or \(V\)

Length of Major Axis: 1\(r\)  
Length of Minor Axis: 6.32

\(c = 7.35\)  
Foci points: \((-7.35, 8)\), \((7.35, 8)\)  
Vertices: \((-8, 8)\), \((8, 8)\)

Endpoints of Minor Axis: \((0, 4.14)\), \((0, 11.16)\)

4. \( \frac{(x-1)^2}{4} + \frac{(y+3)^2}{20} = 1 \)

Center: \((1, -3)\)  
Major Axis: \(H\) or \(V\)

Length of Major Axis: 8.94  
Length of Minor Axis: 4

\(c = 4\)  
Foci points: \((-1, -7)\), \((-1, 1)\)  
Vertices: \((-1, -7.47)\), \((1, 1.47)\)

Endpoints of Minor Axis: \((-1, -3)\), \((3, -3)\)
Part 2: Complete each problem below.

1. A semielliptical arch over a tunnel for a road through a mountain has a width of 60 feet and a maximum height in the center of 28 feet. An oversized load truck traveling through the tunnel is 15 feet tall. What is the maximum allowable width of the truck if it is allowed to occupy both lanes of the tunnel, yet requires a vertical clearance of 1 foot to the top of the tunnel?

\[
\frac{x^2}{900} + \frac{y^2}{784} = 1
\]

2. Hailey's comet has an elliptical orbit with the sun at one focus. The eccentricity (\(e = \frac{c}{a}\)) of the orbit is approximately 0.97. The length of the major axis of the orbit is about 36.23 astronomical units (1 \(au = 93,000,000\) miles). Find an equation for the orbit in miles, using the center of the orbit as the origin and major axis as the \(x\)-axis.

\[
a = \frac{1,084,000,000}{2} = 542,000,000 \text{ miles}
\]

\[
b^2 = a^2 - c^2 = 542,000,000^2 - 1.67 \times 10^7
\]

\[
\frac{x^2}{2.84 \times 10^7} + \frac{y^2}{1.66 \times 10^7} = 1
\]

3. The first artificial satellite to orbit the earth was Sputnik 1 (launched by Russia in 1957). Its highest point above the earth's surface was 938 kilometers and its lowest point was 212 kilometers. The radius of earth is 6,378 kilometers. Find the eccentricity of the orbit (\(e = \frac{c}{a}\)). HINT: The focus of the orbit is the center of the earth.

\[
2a = 2(938) + 212 = 938
\]

\[
a = 469 m
\]

\[
e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{0.953}{a} = 0.953 \text{ km}
\]

\[
c = \sqrt{a^2 - b^2} = \sqrt{6378^2 - 469^2}
\]

\[
\approx 6,317 \text{ km}
\]
Part 3: Write an equation for each ellipse below in standard form. Identify & plot the foci.

1. 

\[
\frac{x^2}{9} + \frac{y^2}{b} = 1
\]

\[
c^2 = a^2 - b^2
\]

\[
9 - b = 2.83
\]

\[
b = 2.83
\]

Foci:

\[
(-2.83, 0) \quad (2.83, 0)
\]

2. 

\[
\frac{(x-2)^2}{9} + \frac{y^2}{1} = 1
\]

Foci:

\[
(0, -2.5) \quad (0, 2.5)
\]

Part 4: Write an equation for each ellipse below in standard form, using the origin as the center.

1. Foci \((\pm 2\sqrt{3}, 0)\); Length of Major Axis = 8

\[
c^2 = a^2 - b^2
\]

\[
(4^2) > 9 = 9 - b^2
\]

\[
12 = 16 - b^2
\]

\[
b = 2
\]

2. Vertices \((0, \pm 6)\); Foci \((0, \pm 2)\)

\[
c^2 = a^2 - b^2
\]

\[
2^2 = (6)^2 - b^2
\]

\[
b = 5.66
\]

Part 5: Graph each ellipse below. Label the center, vertices, foci, and endpoints on the minor axis.

1. \(4x^2 + 3y^2 - 32x + 18y + 43 = 0\)

\[
(4(y^2 + \underline{8}y + \underline{\quad})) + 3(x^2 + 8x + \underline{\quad}) = -43
\]

\[
4(y^2 + 8y + 16) + 3(x^2 + 8x + 16) = -43 + 64 + 48
\]

\[
4(y+4)^2 + 3(x+4)^2 = 48
\]

Center: \((4, -4)\)

Vertices: \((4, -7)\) \((4, 1)\)

Foci: \((4, 5)\) \((4, -1)\)

Minor Axis: \((0, 4, -3)\) \(7.46, -3)\)
Part 5 (con't): Graph each ellipse below. Label the center, vertices, foci, and endpoints on the minor axis.

2. \( 9x^2 + 25y^2 - 36x - 50y - 164 = 0 \)

\[
\left(9\left(x - \frac{2}{9}\right)^2 - \frac{36}{9}\right) + \left(25\left(y - \frac{5}{25}\right)^2 - \frac{50}{25}\right) = \frac{164}{9} + \frac{50}{25}
\]

\[
9\left(x - 2\right)^2 + 25\left(y - 1\right)^2 = 225
\]

\[
\frac{(x-2)^2}{25} + \frac{(y-1)^2}{9} = 1
\]

Center: \((2, 1)\)  
Minor Axis: \((0,0) (4, 2)\)

Vertices: \((-3, 1) \ (7, 1)\)  
Foci: \((-2, 1) \ (6, 1)\)

**Homework**

**Part 1**: Write an equation for each ellipse below in standard form.

1. Vertices \((5, 0), (5, 12)\); Endpoints of Minor Axis \((0, 6), (10, 6)\)

Center: \((5, 0)\)

\[
a = 5 \quad b = 3
\]

\[
\frac{(x-5)^2}{25} + \frac{(y-0)^2}{36} = 1
\]

2. Foci \((1, 5), (8, 5)\); Length of Major Axis = 16

Center: \((4.5, 5)\)

\[
a = 8 \quad c = 3.5
\]

\[
b = \sqrt{7.19}
\]
Part 2: Graph each ellipse below. Label the center, vertices, foci, and end points on the minor axis.

1. \[12x^2 + 20y^2 - 12x + 40y - 37 = 0\]

\[12(x^2 - x + \_ ) + 20(y^2 + 2y + \_ ) = 37\]

\[12(x^2 - x + \frac{1}{4} ) + 20(y^2 + 2y + 1) = 37 + 3 + 20\]

\[12\left(x - \frac{1}{2}\right)^2 + 20\left(y + 1\right)^2 = 60\]

\[\frac{\left(x - \frac{1}{2}\right)^2}{5} + \frac{\left(y + 1\right)^2}{3} = 1\]

Center: \((-\frac{1}{2}, -1)\)  Minor Axis: \((-\frac{1}{2}, 2.73)\)

Vertices: \((-1.74, -1), (2.74, -1)\)  Foci: \((-0.91, -1), (1.91, -1)\)

Part 3: Complete the problem below.

1. The arc of a tunnel is semi-elliptical with the major axis horizontal. The base of the tunnel is 9 meters wide and the maximum height of the tunnel is 3 meters. If two support columns extend vertically from the foci to the ceiling of the tunnel overhead, calculate the height of each support column.

\[c^2 = a^2 - b^2\]

\[= 4.5^2 - 3^2\]

\[= 20.25 - 9\]

\[c = 3.354 \text{ m}\]

\[\frac{x^2}{20.25} + \frac{y^2}{4} = 1\]

so \(x = 3.354\)

\[\frac{3.354^2}{20.25} + \frac{y^2}{4} = 1\]

\[\frac{y^2}{4} = 0.4\]

\[y^2 = 4\]

\[y = 2 \text{ m. tall}\]
Check for Understanding

Directions: Write everything you remember about circles, parabolas, and ellipses.