6.4 Vectors & Dot Products

Standard: N-VM #3: Solve problems involving velocity and other quantities that can be represented by vectors.

Essential Question: How can we use vectors to solve problems?

Daily Question: Given \( \mathbf{u} = (-3, 7) \) and \( \mathbf{v} = (4, 6) \), find each of the following...

1. \( \mathbf{u} - 2\mathbf{v} \)
\[ (\mathbf{u} - 2\mathbf{v}) = (-3, 7) - 2(4, 6) = (-3, 7) - (8, 12) = (-11, -5) \]

2. \( 3\mathbf{u} + \mathbf{v} \)
\[ (3\mathbf{u} + \mathbf{v}) = 3(-3, 7) + (4, 6) = (-9, 21) + (4, 6) = (-5, 27) \]

3. \( -2\mathbf{u} + 4\mathbf{v} \)
\[ (-2\mathbf{u} + 4\mathbf{v}) = -2(-3, 7) + 4(4, 6) = (6, -14) + (16, 24) = (22, 10) \]

Dot Product

Dot Products are used to determine the angle between two vectors, derive geometric theorems, and solve physics problems (mechanical force & magnetic flux). The product yields a scalar, rather than a vector.

In physics, the dot product is used to calculate mechanical force and magnetic flux.

The dot product of two vectors \( \mathbf{u} = (a, b) \) and \( \mathbf{v} = (c, d) \) is denoted by \( \mathbf{u} \cdot \mathbf{v} \), read “\( \mathbf{u} \) dot \( \mathbf{v} \),” given by

\[ \mathbf{u} \cdot \mathbf{v} = ac + bd \]

Part 1: Find the dot product.

1. \( \langle 2, 3 \rangle \cdot \langle 4, -1 \rangle \)
\[ \langle 2, 3 \rangle \cdot \langle 4, -1 \rangle = 8 - 3 = 5 \]

2. \( \langle 6, 4 \rangle \cdot \langle -2, 3 \rangle \)
\[ \langle 6, 4 \rangle \cdot \langle -2, 3 \rangle = -12 + 12 = 0 \]

3. \( \langle -5, 1 \rangle \cdot \langle 3, 7 \rangle \)
\[ \langle -5, 1 \rangle \cdot \langle 3, 7 \rangle = -15 + 7 = -8 \]

Properties of the Dot Product

Let \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \) be scalars in a place and let \( c \) be a scalar. Then:

1. \( \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \)
2. \( 0 \cdot \mathbf{v} = 0 \)
3. \( \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w} \)
4. \( \mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2 \)
5. \( c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot cv \)
**Angle Between Two Vectors**

\[ \| v - u \| ^2 = \| v \| ^2 + \| u \| ^2 - 2 \| u \| \| v \| \cos \theta \]

\[(v - u) \cdot (v - u) = \| v \| ^2 + \| u \| ^2 - 2 \| u \| \| v \| \cos \theta \]

\[ \| v \| ^2 - 2u \cdot v + \| u \| ^2 = \| v \| ^2 + \| u \| ^2 - 2 \| u \| \| v \| \cos \theta \]

\[ -2u \cdot v = -2 \| u \| \| v \| \cos \theta \]

\[-2u \cdot v \]

\[ \cos \theta = \frac{u \cdot v}{\| u \| \| v \|} \]

**Part 2: Draw each vector below. Then, find the angle between the two vectors.**

1. \( u = (3, 4), \ v = (2, 1) \)

   \[
   \cos \theta = \frac{6 + 4}{5 \cdot \sqrt{5}}
   \]

   \[ \theta = 60.57^\circ \]

2. \( u = (1, 6), \ v = (7, 2) \)

   \[
   \cos \theta = \frac{7 + 12}{\sqrt{37} \cdot \sqrt{53}}
   \]

   \[ \theta = 64.59^\circ \]

**Homework**

**Part 1: Find the dot product for each pair of vectors.**

1. \( (6, -1), \ (2, 5) \)

   \[ u \cdot v = 12 - 5 = 7 \]

2. \( (2, 4), \ (0, -1) \)

   \[ u \cdot v = 0 - 4 = -4 \]

**Part 2: Find the angle between each pair of vectors.**

1. \( (2, 1), \ (-3, 1) \)

   \[
   \cos \theta = \frac{-6 + 1}{\sqrt{5} \cdot \sqrt{5}}
   \]

   \[ \theta = 135^\circ \]

2. \( (-5, 12), \ (3, 2) \)

   \[
   \cos \theta = \frac{-15 + 24}{13 \cdot \sqrt{13}}
   \]

   \[ \theta = 101.07^\circ \]