10.1 – 10.3 Conic Sections

**Standard:** G-GPE – Translate between the geometric description and the equation for a conic section.

**Essential Question:** How are conic sections formed?

**Directions:**

1. Use paper and tape to create a double-napped cone. Then, using scissors, cut the cone as per each diagram below. After each cut, ask yourself, what shape is created by the cross-section on the surface of the cone? Write your answer on the space provided below.

<table>
<thead>
<tr>
<th>Cut #1</th>
<th>Cut #2</th>
<th>Cut #3</th>
<th>Cut #4</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Cut here →" /></td>
<td><img src="image2" alt="Cut here →" /></td>
<td><img src="image3" alt="Cut here →" /></td>
<td><img src="image4" alt="Cut here →" /></td>
</tr>
</tbody>
</table>

**Shape created:**

- Cut #1
- Cut #2
- Cut #3
- Cut #4


**Conic section** – the intersection of a plane and a double-napped cone.

**General Form Equation of a Conic Section:**

\[ Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \]
In order to accurately graph conic sections, we must be comfortable completing the square of a second-degree equation.

**Part 1:** Solve each equation below by completing the square.

1. \( x^2 - 16x + 1 = 0 \)
2. \( 2x^2 + 12x + 14 = 0 \)
3. \( 3x^2 + 4x + 6 = x^2 + 6x \)
4. \( 5x^2 + 2x + 3 = 10 - 8x \)

**Problem:** You are designing a city park like the one shown in the picture. You want the park to have two fountains that are equidistant from four of the six park entrances. Calculate the distance the each fountain is, in yards, from the nearest four entrances.
CIRCLES

**Standard:** G-GPE #1 – Derive the equation for a circle given the center and radius.

**Essential Question:** How do we write the equation of a circle?

**Circle** – the collection of all points \((x, y)\) that are equidistant from a fixed point \((h, k)\)

**Standard Form Equation of a Circle:**

\[(x - h)^2 + (y - k)^2 = r^2\]

**Part 1:** Identify the center and radius of each circle below.

1. \((x + 4)^2 + (y - 2)^2 = 25\)  
2. \((x - 1)^2 + (y + 3)^2 = 32\)  
3. \((x + 5.5)^2 + (y + 1)^2 = 45\)

**Part 2:** Identify the radius and center (if applicable) in each scenario below.

1. Center at \((-2, 7)\); passes through the origin  
2. Diameter has its endpoints on \((7, -5)\) and \((-6, 4)\)

3. Center at \((-3, 4)\); tangent to the x-axis  
4. Center at \((3, 4)\); passes through \((5, 6)\)

*NOW*, write the equation of each circle above in standard form.
*Converting a general form circle equation into a standard form circle equation.

<table>
<thead>
<tr>
<th>Steps</th>
<th>Example: $2x^2 + 2y^2 - 4x - 12y + 8 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Group $x$-terms and $y$-terms and solve for the constant</td>
<td>$(2x^2 - 4x) + (2y^2 - 12y) = -8$</td>
</tr>
<tr>
<td>2. Get the leading coefficient of the square terms equal to 1</td>
<td>$(x^2 - 2x) + (y^2 - 6y) = -4$</td>
</tr>
<tr>
<td>3. Complete the square for $x$ and $y$ values. Keep the equation BALANCED!</td>
<td>$(x^2 - 2x + 1) + (y^2 - 6y + 9) = -4 + 1 + 9$</td>
</tr>
<tr>
<td>4. Factor each trinomial and write in standard form</td>
<td>$(x - 1)^2 + (y - 3)^2 = 6$ Center $(1, 3); \text{radius } \sqrt{6}$</td>
</tr>
</tbody>
</table>

Part 3: Convert each circle from general to standard form. Identify the center of the circle, the radius, and sketch a graph of each circle.

1. $x^2 + y^2 + 4x + 10y = 7$

2. $4x^2 + 4y^2 - 16x + 24y + 16 = 0$

3. $y^2 - 2y = 24 - x^2$

4. $x^2 - 10y = 1 - 20x - y^2$

NOW, check the graphs of your circles above by graphing the functions using your graphing calculator.
Homework

Part 1: Write the equation of each circle below in standard form.

1. Center $(6, -3)$; tangent to the $y$-axis
2. Center $(0, 2)$; radius $= 3$

2. Center $(5, -2)$; radius $= \sqrt{61}$
4. Center $(-2, -7)$; tangent to $x$-axis

Part 2: Convert each circle from general to standard form. Identify the center of the circle and the radius.

1. $x^2 + y^2 + 14y = 0$
2. $4x^2 + 4y^2 - 4x - 8y - 59 = 0$

3. $10x^2 + 10y^2 = 80x$
Parabolas

**Standard:** G-GPE #2 – Derive the equation for a parabola given a focus and directrix.

**Essential Question:** How do we write the equation of a parabola?

In Algebra II, you learned that parabolas can open up or down using the equation \( y = ax^2 + bx + c \). The definition above and standard form below are generalized for parabolas that open up, down, left, and right.

**Parabola** – a curve on which any point is equidistant from the **directrix** and **focus**
- **Focus** – a point located on the axis of symmetry
- **Directrix** – a line perpendicular to the axis of symmetry
- **Vertex** – the midpoint between the focus and directrix
- **Latus Rectum** – a chord parallel to the directrix that goes through the focus (its endpoints are on the graph)

**Standard Form Equation of a Parabola:** with vertex \((h, k)\) and \(p = \text{distance from vertex to the focus}\)

**Vertical Axis of Symmetry**
\[
(x - h)^2 = 4p(y - k)
\]

**Horizontal Axis of Symmetry**
\[
(y - k)^2 = 4p(x - h)
\]

*The sign of \(p\) determines whether the parabola opens up, down, left, or right.*

Where do parabolas exist?

- **Headlights**
- **Satellite Dishes**
- **Solar Cookers**
- **Telescopes with parabolic mirrors**
- **Cables of Suspension Bridges**
Part 1: Complete each problem below.

1. Write the equation of the parabola in standard form with vertex $(2, 1)$ and focus $(2, 4)$. Graph the vertex, focus, latus rectum, directrix, and parabola.
   
   **Open**: Up Down Left Right
   
   - Distance from vertex to focus $(p)$: 
   - Directrix: 
   - Length of latus rectum $(4p)$: 
   - Standard Form Equation: 

2. Write the equation of the parabola in standard form with vertex $(4, 2)$ and directrix $y = 5$. Graph the vertex, focus, latus rectum, directrix, and parabola.
   
   **Open**: Up Down Left Right
   
   - Distance from vertex to focus $(p)$: 
   - Directrix: 
   - Length of latus rectum $(4p)$: 
   - Standard Form Equation: 

3. Write the equation of the parabola in standard form with vertex $(-1, 3)$ and focus $(-3, 3)$. Graph the vertex, focus, latus rectum, directrix, and parabola.
   
   **Open**: Up Down Left Right
   
   - Distance from vertex to focus $(p)$: 
   - Directrix: 
   - Length of latus rectum $(4p)$: 
   - Standard Form Equation: 
4. Write the equation of the parabola in standard form with focus 
(2,2) and directrix $x = -3$. Graph the vertex, focus, latus 
rectum, directrix, and parabola.

**Open:** Up Down Left Right

**Distance from vertex to focus ($p$):**

**Directrix:**

**Length of latus rectum ($4p$):**

**Standard Form Equation:**

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**Part 2:** Complete each problem below.

1. Each cable of the Golden Gate Bridge is suspended (in the shape of a parabola) between two towers that are 4,200 feet apart. The top of each tower measures 500 feet above the roadway and the cables touch the roadway midway between the towers. Find an equation for the parabolic shape of each cable.

   **HINT:** Draw a diagram for the bridge.

2. The receiver (focus) in a parabolic television dish antenna is 3.5 feet above the vertex. Find an equation of a cross-section of the reflector, assuming that the dish is directed upward and that the vertex is at the origin.
Part 3: Graph each parabola below. Identify the vertex, focus, directrix, and length of the latus rectum.

1. \((x + 5)^2 = -2(y - 1)\)

2. \(y^2 = -8x\)

Vertex: \(p:\)

Directrix: \(p:\)

Length of latus rectum \((4p)\):

3. \(3(y + 2)^2 = -36x\)

4. \(x^2 - 4x = 3y + 11\)

Vertex: \(p:\)

Directrix: \(p:\)

Length of latus rectum \((4p)\)
Homework

Part 1: What is the difference between a catenary and a parabola? Provide three examples of catenaries.

Part 2: Given the following, find the equation of the parabola.

1. Vertex (1, -3); Focus (-2, -3)  
2. Focus (5, 1); Directrix y = -3

Part 3: Complete the problem below.

1. A baseball is thrown from a height of 6 feet by an outfielder to the catcher at home plate. By discounting air resistance, the path of the baseball can be modeled by a parabola. After traveling 135 feet horizontally, the ball reaches its maximum height of 25 feet. How far did the outfielder throw the ball if the catcher catches the ball at a height of 1 foot? Round your answer to the nearest foot.
Ellipses

**Standard:** G-GPE #3 – Derive the equation of ellipses given the foci.

**Essential Question:** How do we write the equation of an ellipse?

**Ellipse** – the set of all points \((x, y)\) whose sum of the distances from two distinct points, called **foci**, is constant

**Vertices** – where the line through the foci intersects the ellipse

**Major axis** – the chord joining the vertices

**Minor axis** – the chord perpendicular to the major axis at the center

**Center** – the intersection of the major and minor axis

**Standard Form Equation of an Ellipse:** with **center** \((h, k)\), **major axis** of length \(2a\), & **minor axis** of length \(2b\), where \(0 < b < a\). The **foci** lie on the major axis \(c\) units away from the center, where \(c^2 = a^2 - b^2\).

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1
\]

*The larger denominator (\(a\)) determines whether the major axis is horizontal or vertical.*

**Example**

1. \[
\frac{(x-5)^2}{16} + \frac{(y+4)^2}{9} = 1
\]

  - **Center:**
  - **Major Axis:** H or V
  - **Length of Major Axis:**
  - **Length of Minor Axis:**
  - **Foci points:**
  - **Vertices:**
  - **Endpoints of Minor Axis:**
Where do ellipses exist?

Orbits of Planets

Whispering rooms - Video

Lithotripter

Architecture

Elliptical billiard table

Part 1: Graph each ellipse below.

1. \[ \frac{(x+1)^2}{4} + \frac{(y-4)^2}{25} = 1 \]

- Center:
- Major Axis: H or V
- Length of Major Axis:
- Length of Minor Axis:
- Foci points:
- Vertices:
- Endpoints of Minor Axis:
2. \( \frac{(x-3)^2}{16} + (y + 2)^2 = 1 \)

- **Center:** 
- **Major Axis:** H or V
- **Length of Major Axis:** 
- **Length of Minor Axis:** 
- **Foci points:** 
- **Vertices:**
- **Endpoints of Minor Axis:**

3. \( \frac{x^2}{64} + \frac{(y-8)^2}{10} = 1 \)

- **Center:** 
- **Major Axis:** H or V
- **Length of Major Axis:** 
- **Length of Minor Axis:** 
- **Foci points:** 
- **Vertices:**
- **Endpoints of Minor Axis:**

4. \( \frac{(x-1)^2}{4} + \frac{(y+3)^2}{20} = 1 \)

- **Center:** 
- **Major Axis:** H or V
- **Length of Major Axis:** 
- **Length of Minor Axis:** 
- **Foci points:** 
- **Vertices:**
- **Endpoints of Minor Axis:**
Part 2: Complete each problem below.

1. A semielliptical arch over a tunnel for a road through a mountain has a width of 60 feet and a maximum height in the center of 28 feet. An oversized load truck traveling through the tunnel is 15 feet tall. What is the maximum allowable width of the truck if it is allowed to occupy both lanes of the tunnel, yet requires a vertical clearance of 1 foot to the top of the tunnel?

2. Hailey’s comet has an elliptical orbit with the sun at one focus. The eccentricity \(e = \frac{c}{a}\) of the orbit is approximately 0.97. The length of the major axis of the orbit is about 36.23 astronomical units (1 au = 93,000,000 miles). Find an equation for the orbit in miles, using the center of the orbit as the origin and major axis as the x-axis.

3. The first artificial satellite to orbit the earth was Sputnik 1 (launched by Russia in 1957). Its highest point above the earth’s surface was 938 kilometers and its lowest point was 212 kilometers. The radius of earth is 6,378 kilometers. Find the eccentricity of the orbit \(e = \frac{c}{a}\). HINT: The focus of the orbit is the center of the earth.
Part 3: Write an equation for each ellipse below in standard form. Identify & plot the foci.

1. 

2. 

Part 4: Write an equation for each ellipse below in standard form, using the origin as the center.

1. Foci \((±2\sqrt{3}, 0)\); Length of Major Axis = 8

2. Vertices \((0,±6)\); Foci \((0,±3)\)

Part 5: Graph each ellipse below. Label the center, vertices, foci, and endpoints on the minor axis.

1. \(4x^2 + 3y^2 − 32x + 18y + 43 = 0\)
Part 5 (con’t): Graph each ellipse below. Label the center, vertices, foci, and endpoints on the minor axis.

2. \(9x^2 + 25y^2 - 36x - 50y - 164 = 0\)

Homework

Part 1: Write an equation for each ellipse below in standard form.

1. Vertices (5,0), (5, 12); Endpoints of Minor Axis (0,6), (10,6)

2. Foci (1,5), (8, 5); Length of Major Axis = 16
Part 2: Graph each ellipse below. Label the center, vertices, foci, and end points on the minor axis.

1. \[12x^2 + 20y^2 - 12x + 40y - 37 = 0\]

Part 3: Complete the problem below.

1. The arc of a tunnel is semi-elliptical with the major axis horizontal. The base of the tunnel is 9 meters wide and the maximum height of the tunnel is 3 meters. If two support columns extend vertically from the foci to the ceiling of the tunnel overhead, calculate the height of each support column.
Check for Understanding

Directions: Write everything you remember about circles, parabolas, and ellipses.
Hyperbolas

**Standard:** G-GPE #3 – Derive the equation of hyperbolas given the foci.

**Essential Question:** How do we write the equation of a hyperbola?

Hyperbola – the set of all points \((x, y)\) such that the difference of the distances from two distinct fixed points, called foci, is a positive constant

- **Transverse axis** – the line segment connecting the vertices
- **Conjugate axis** – the axis perpendicular to the transverse axis (not shown)

The graph of a hyperbola consists of two disconnected branches

**Standard Form Equation of a Hyperbola:** with center \((h, k)\)

<table>
<thead>
<tr>
<th>Transverse Axis of Horizontal</th>
<th>Transverse Axis of Vertical</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1)</td>
<td>(\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1)</td>
</tr>
</tbody>
</table>

*The vertices are \(a\) units from the center, and the foci are \(c\) units from the center, where \((c^2 = a^2 + b^2)\).*

Where do hyperbolas exist?

- **Sonic Boom Curve**
- **LORAN**
- **Architecture**
- **Cooling Towers**
- **Skis / Snowboards**
Directions: Using the equation below and graph to the right, locate and calculate the important aspects of a hyperbola.

\[
\frac{y^2}{4} - \frac{(x+1)^2}{3} = 1
\]

Center: Transverse Axis: H or V

Vertices: Foci:

Asymptote Slopes:

Part 1: Graph each hyperbola below.

1. \[
\frac{x^2}{64} - \frac{(y-8)^2}{25} = 1
\]

Center: Transverse Axis: H or V

Vertices: Foci:

Asymptote Slopes:

2. \[
\frac{(y-2)^2}{8} - \frac{(x+3)^2}{16} = 1
\]

Center: Transverse Axis: H or V

Vertices: Foci:

Asymptote Slopes:
Part 2: Find the equation of each hyperbola below. Then, graph each hyperbola.

1. Vertices $(\pm 2, 2)$; Foci $(\pm 4, 2)$

2. Vertices $(0, \pm 3)$; Asymptotes $y = \pm 3x$

3. Vertices $(\pm 4, 0)$; Foci $(\pm 6, 0)$
Part 3: Convert each hyperbola to standard form. Then, graph each hyperbola.

1. \(9x^2 - y^2 - 36x - 6y + 18 = 0\)

2. \(-x^2 + 9y^2 - 2x + 72y + 107 = 0\)

Homework

Part 1: Write the equation of the hyperbola in standard form. Then, graph the hyperbola.

1. Vertices \((3, 0), (3, 4)\); Asymptotes \(y = \frac{2}{3}x, y = 4 - \frac{2}{3}x\)
*THINK: Look closely at the problems on page # 4, 9, 15/16, 22. How can we determine what type of conic function exists simply by looking at the conic equation in general form? Use previous problems to assist you.

**Directions:** Identify each conic section below.

1. \( x^2 + 4x + 6y - 2 = 0 \)  
2. \( 6x^2 + 2y^2 + 18x - 10y + 2 \)  
3. \( 3y^2 + 3x^2 - 4x - 4 = 0 \)  
4. \( y^2 - 8x - 2y - 17 = 0 \)  
5. \( x^2 - y^2 - 4x + 6y - 3 = 0 \)  
6. \( 9x^2 - y^2 - 36x - 6y + 18 = 0 \)  
7. \( x^2 + 4y^2 - 6x + 20y - 2 = 0 \)
Conic Sections RECAP

Standard Form Equations: where \((h, k)\) is the vertex of a parabola & the center of other conics

Circle:

Parabola (opens ↑ or ↓):

Parabola (opens ← or →):

Horizontal Axis

Vertical Axis

Ellipse:

Hyperbola:

Directions: Convert the following to standard form and graph.

1. \(x^2 + y^2 - 14x + 4y - 11 = 0\)

2. \(x^2 + 4y^2 - 10x + 16y + 37 = 0\)
Directions (con’t): Convert the following to standard form and graph.

3. \(9y^2 - x^2 - 54y + 8x + 56 = 0\)

4. \(9x^2 + 4y^2 - 36x - 24y + 36 = 0\)

5. \(y^2 + 6y + 8x + 25 = 0\)
Conic Word Problems

Directions: Complete each problem below.

1. The arch of a bridge is in the form of half of an ellipse, with a horizontal major axis. The span of the arch is 12 meters and the height of the arch above the water at the center is 4 meters. Using the origin as the center of the semi-ellipse, how high above the water is the arch at a point on the water 2 meters from one of the ends of the arch? Round our answer to the nearest hundredth.

2. A solar collector is a concave mirror that concentrates the rays of the sun to produce high temperatures. In the context of buildings, the absorbed energy typically heats water, which is then used for space heating and/or domestic hot water. The distance from the vertex to the focus is 6 feet. The rotatable solar collector will be placed on the rooftop at dawn as shown, with the vertex at (0, 0). Write an equation, in standard form, for the parabola that models the solar collector and its location on the roof. What is the equation of the directrix and axis of symmetry?

3. A room with an elliptical ceiling forms a “whispering gallery,” as shown below. Thanks to the reflective property of the ellipse, a whispered message at one focus can be heard clearly by someone standing across the room at the other focus. If the elliptical ceiling has a major axis of 120 feet and a minor axis of 72 feet, how far apart are the people? What is the equation of the ellipse?